Financial options

1. Basic principles

a. Definition

An option is a right to buy (call) or to sell (put) an underlying asset, on the expiration date (European option) or during a given period (American option) at a price which is known in advance (Strike price).

The value of the option is the premium

b. Premium breakdown

The premium has to components:

- Intrinsic value (IV)
  - For a call: IV = max(0;S-E)
    - E= strike price: 100€; S = Spot price of the underlying asset : 120€; IV=120-100 = 20€
    - E=100€; S=90€; IV = 0
  - For a put: IV = max(0;E-S)
    - E= strike price : 100€; S = Spot price of the underlying asset : 120€; IV = 0€
    - E=100€; S=90€; IV = 100 – 90 = 10€
- Time premium or time value (TV) which is the additional amount, beyond the IV, the trader is ready to pay given his expectations regarding the price of the underlying asset until the expiration date of the option.

On option's expiration date, there is no possible hope regarding a favourable evolution of the underlying asset's price evolution.

Then, on expiration date:

- TV = 0
- Premium= IV

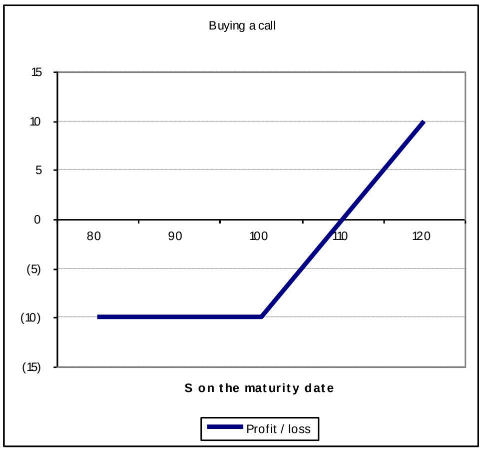
2. Customary speculative strategies

a. Buying a call

Assumptions:

- Strike price: 100 €
- Premium: 10 €

S on the maturity date	80	90	100	110	120
Premium paid	(10)	(10)	(10)	(10)	(10)
Intrinsic Value	0	0	0	10	20
Profit / loss	(10)	(10)	(10)	0	10



The potential losses are limited to the premium which has been paid, whereas the gains are the more important as the price of the underlying asset is high.

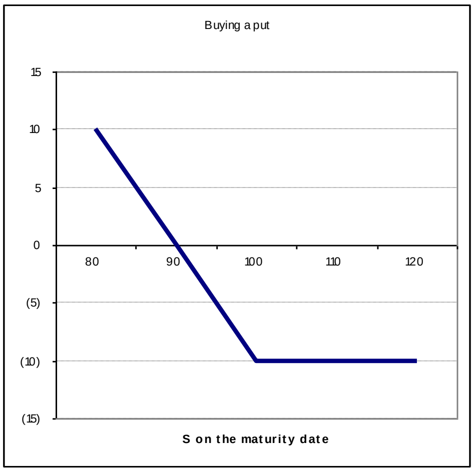
It's therefore a strategy based on expectations on the increase in the value of the underlying asset.

b. Buying a put

Assumptions:

- Strike price: 100 €
- Premium: 10 €

S on the maturity date	80	90	100	110	120
Premium paid	(10)	(10)	(10)	(10)	(10)
Intrinsic Value	20	10	0	0	0
Profit / loss	10	0	(10)	(10)	(10)



The potential losses are limited to the premium which has been paid whereas the gains are the more important as the spot price of the underlying asset is low. This speculative strategy corresponds to expectations on the decrease in the spot price of the underlying asset.

c. Selling a call

Assumptions:

- Strike price: 100 €
- Premium: 10 €

Selling a call					
S on the maturity date	80	90	100	110	120
Premium received	10	10	10	10	10
Financing of the intrinsic value	0	0	0	(10)	(20)
Profit / loss	10	10	10	0	(10)

If the spot price of the underlying asset is 110€, the call with a strike price of 100€ is interesting; therefore the owner of such a call decides to exercise it, but as a trader on options, the seller of the call does not own the underlying asset: he has to buy it on the stock market for a consideration corresponding to its market price (110€); then he sells it to his counterpart for 100€ and books a 10€ capital loss.

The gains are limited to the premium which has been received (10€) whereas the losses are the more important as the spot price of the underlying asset is high. It is a strategy coressponding to expectations on the decrease in the price of the underlying asset. This strategy is more risky and has lower profitable prospects than the purchase of a put. The sale of a call is however achieved by the trader who speculates on the decrease of the underlying asset while refusing to pay a premium.

d. Selling a put

Assumptions:

- Strike price: 100 €
- Premium: 10 €

S on the maturity date	80	90	100	110	120
Premium received	10	10	10	10	10
Financing of the intrinsic value	(20)	(10)	0	0	0
Profit / loss	(10)	0	10	10	10

If the spot price of the underlying asset is 80€, the put with the strike price of 100€ is interresting; then its owner decides to exercise it and asks the seller of the put to buy the underlying asset. Then, the counterpart of the seller of the put sells the underlying assets to the seller of the put for a consideration of 100€. As the seller of the put can not keep the underlying asset in his securities portfolio, he has to sell it on the stock market for a consideration of 80€ and books a 20€ capital loss.

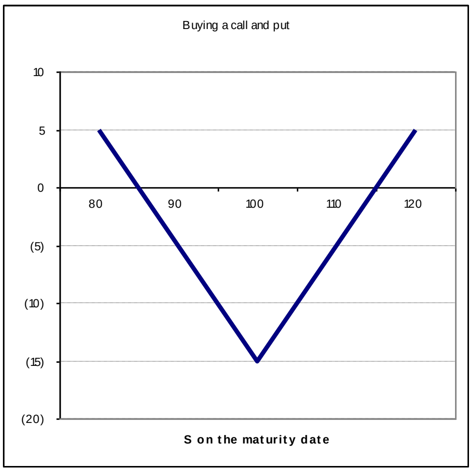
The gains are limited to the premium which has been received (10€) whereas the losses are the more important as the spot price of the underlying asset is low; then it is a strategy based on expectations on the increase of the price of the underlying asset.

e. Buying a call and a put with the same strike price

Assumptions:

- Strike price: 100 €
- Premium:
  - Call: 10 €
  - Put: 5 €

S on the maturity date	80	90	100	110	120
Premiums paid	(15)	(15)	(15)	(15)	(15)
Intrinsic Value of the call	0	0	0	10	20
Intrinsic Value of the put	20	10	0	0	0
Profit / loss	5	(5)	(15)	(5)	5



The highest loss occurs when the spot price of the underlying asset equals the strike price of the option; this maximum loss corresponds to the premiums which have been initially paid.

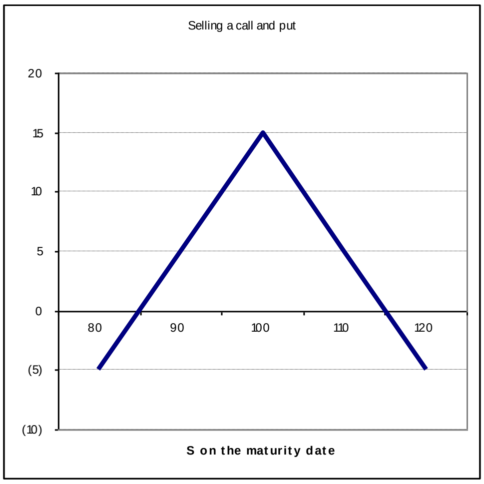
The gains are the more important as the increase or the decrease is significant; this strategy corresponds to an expectation on volatility of the underlying asset.

f. Selling a call and a put with the same strike price

Assumptions:

- Strike price: 100 €
- Premium:
    - Call: 10 €
    - Put: 5 €

S on the maturity date	80	90	100	110	120
Premiums received	15	15	15	15	15
Financing of the intrinsic value of the call	0	0	0	(10)	(20)
Financing of the intrinsic value of the put	(20)	(10)	0	0	0
Profit / loss	(5)	5	15	5	(5)



The maximum gain (15€) occurs when the spot price of the underlying asset equals the strike price; this gain corresponds to the premium which has been received.

The losses are the more important as the volatility is high; therefore this strategy corresponds to expectations on stability of the underlying asset.

g. Butterfly

Buying a call, Premium C1, Strike price E1
Selling 2 calls, Premium by call C2, Strike price E2
Buying a call, Premium C3, Strike price E3

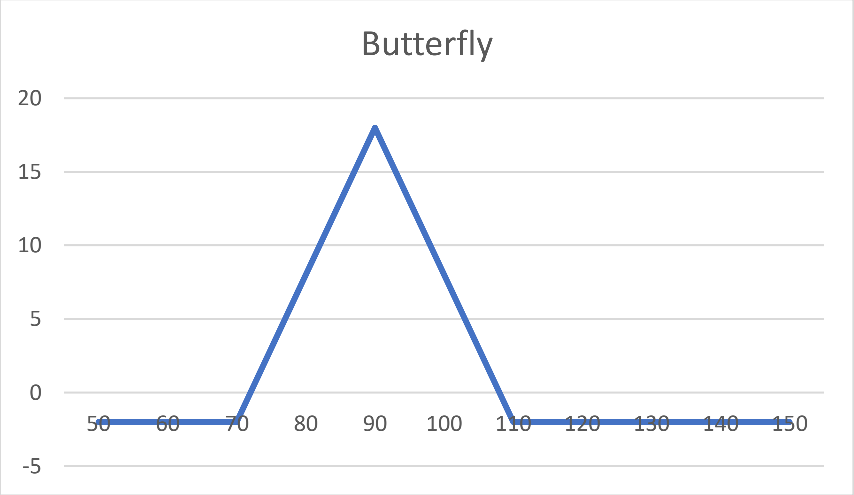
E1<E2<E3 and E2=(E1+E3)/2

As long as the spot price S of the underlying asset has not reached E2, only the 1st call has intrinsic value. The pretax profit is therefore that of one call, taking into account the algebraic sum of premiums paid and received.

When S has reached E2, the 2 calls that have been sold have some intrinsic value. That of one of the 2 sold calls is compensated by that of the call which gas been purchased. Then once S is higher than E2, the evolution of the pretax profit is in line with that of one sold call.

Finally, once S is higher than E3, the 4 options have intrinsic value. The intrinsic value of the 2 purchased calls is compensated by that of the 2 sold calls. The the pretax profit corresponds to the algebraic sum of the premiums paid and received.

C1	11				E1	70					
C2	12				E2	90					
C3	15				E3	110					
S	50	60	70	80	90	100	110	120	130	140	150
Premium	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
IV	0	0	0	10	20	10	0	0	0	0	0
Pretax profit	-2	-2	-2	8	18	8	-2	-2	-2	-2	-2



With such a strategy, the trader anticipates a stability of the price of the underlying asset aroud the strike price of the sold calls (90€).

The combination of positions enables to limit the potential loss to the algebraic sum of the premiums paid and receved.

h. Condor

Buying a call, Premium C1, Strike price E1
Selling a call, Premium by call C2, Strike price E2
Selling a call, Premium by call C3, Strike price E3
Buying a call, Premium C4, Strike price E4

E1<E2<E3<E4 ; E2=(E1+E3)/2 ; E3=(2+E4)/2

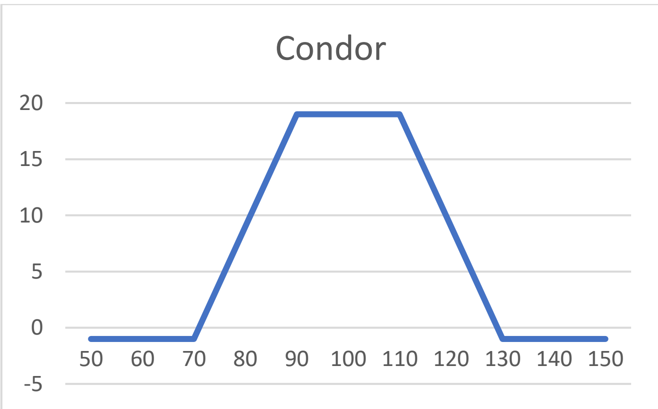
As long as the spot price S of the underlying asset has not reached E2, only the 1^{st} call has intrinsic value. The pretax profit is therefore that of one call, taking into account the algebraic sum of premiums paid and received.

When S has reached E2, the call that has been sold has some intrinsic value. It is compensated by that of the call which gas been purchased. Then there is no more incremental intrinsic value.

When S has reached E3, the 2 calls that have been sold have some intrinsic value. That of one of the 2 sold calls is compensated by that of the call which gas been purchased. Then once S is higher than E3, the evolution of the pretax profit is in line with that of one sold call.

Finally, once S si higher than E4, the 4 options have intrinsic value. The intrinsic value of the 2 purchased calls is compensated by that of the 2 sold calls. The the pretax profit corresponds to the algebraic sum of the premiums paid and received.

C1	11				E1	70					
C2	12				E2	90					
C3	15				E3	110					
C4	17				E4	130					
S	50	60	70	80	90	100	110	120	130	140	150
Premium	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
IV	0	0	0	10	20	20	20	10	0	0	0
Pretax profit	-1	-1	-1	9	19	19	19	9	-1	-1	-1



With such a strategy, the trader anticipates a stability of the price of the underlying asset between the strike prices of the 2 sold calls (90€ and 110€).

The combination of positions enables to limit the potential loss to the algebraic sum of the premiums paid and receved.

3. Arbitrages

An arbitrage consists in combining positions on 2 markets, as options and future markets, in order to get a foreseeable profit.

In order to be foreseeable, this profit has to be independent from the spot price S of the underlying asset.

a. Reverse conversion

Definition of the reverse conversion: Synthetic purchase + Sale of future

Synthetic purchase = buying a call (premium=C) and selling a put (premium=P) with the same strike price: E

Future contract price: F

Spot price of the underlying asset on expiration date: S

Call, put and future with the same underlying asset

t = time to expiration

r = risk free rate as there is no uncertainty on the future profit.

PBT = profit before tax on expiration date of the 3 derivatives

$$PBT = (-C + P)(1 + r)^t + \max(0, S - E) - \max(0, E - S) + F - S$$

– If $S < E$, the call has no intrinsic value and that of the put is $E - S$.

Then:

$$PBT = (-C + P)(1 + r)^t + 0 - (E - S) + F - S$$

$$PBT = (-C + P)(1 + r)^t - E + F$$

– If $S > E$, the call has intrinsic value that is S-E and the put has none.

Then:

$$PBT = (-C + P)(1 + r)^t + S - E - 0 + F - S$$

$$PBT = (-C + P)(1 + r)^t - E + F$$

In both cases, the PBT does not depend on S and is the unique.

In continuous time:

$$PBT = (-C + P)e^{it} - E + F$$

Where i is the nominal rate for continuous payments and r is the yield to maturity, i and r being equivalent rates.

b. Conversion

Definition of the conversion: Synthetic sale + purchase of future

Synthetic sale = selling a call (premium=*C*) and buying a put (premium=*P*) with the same strike price: E

Future price: F

Call, put and future with the same underlying asset

t = time to expiration

r = risk free rate as there is no uncertainty on the Reverse conversion

Definition of the reverse conversion: Synthetic purchase + Sale of future

Synthetic purchase = buying a call (premium=*C*) and selling a put (premium=*P*) with the same strike price: E

Future contract price: *F*

Spot price of the underlying asset on expiration date: *S*

Call, put and future with the same underlying asset

t = time to expiration

r = risk free rate as there is no uncertainty on the future profit.

PBT = profit before tax on expiration date of the 3 derivatives

$$PBT = (C - P)(1 + r)^t - \max(0, S - E) + \max(0, E - S) + F - S$$

- If $S < E$, the call has no intrinsic value and that of the put is $E - S$. Then:

  $PBT = (C - P)(1 + r)^t - 0 + (E - S) - F + S$

  $PBT = (C - P)(1 + r)^t + E - F$

- If $S > E$, the call has intrinsic value what is *S*-*E* and the put has none. Then:

  $PBT = (C - P)(1 + r)^t - (S - E) + 0 - F + S$

  $PBT = (C - P)(1 + r)^t + E - F$

In both cases, the *PBT* does not depend on *S* and is the unique.

c. Box spread

Definition of the box spread: Synthetic purchase+ synthetic sale

Synthetic purchase = buying a call (premium=*C1*) and selling a put (premium=*P1*) with the same strike price: E1

Synthetic sale = selling a call (premium=*C2*) and buying a put (premium=*P2*) with the same strike price: E2

E1<E2

Calls and puts with the same underlying asset

t = time to expiration

r = risk free rate as there is no uncertainty on the box spread

PBT = profit before tax on expiration date of the 4 derivatives

$$PBT = (-C1 + P1 + C2 - P2)(1+r)^t + \max(0, S - E1) - \max(0, E1 - S) - \max(0, S - E2) + \max(0, E2 - S)$$

- If $S < E1$, the calls have no intrinsic value and the puts have some. Then:

  $$PBT = (-C1 + P1 + C2 - P2)(1+r)^t + 0 - (E1 - S) - 0 + E2 - S$$

  $$PBT = (-C1 + P1 + C2 - P2)(1+r)^t + \mathrm{E2} - \mathrm{E1}$$

- If $E1 < S < E2$, only the 1st call and the second put have some intrinsic value. Then:

  $$PBT = (-C1 + P1 + C2 - P2)(1+r)^t + S - E1 - 0 - 0 + E2 - S$$

  $$PBT = (-C1 + P1 + C2 - P2)(1+r)^t + \mathrm{E2} - \mathrm{E1}$$

- If $S > E2$, only the calls have some intrinsic value. Then:

  $$PBT = (-C1 + P1 + C2 - P2)(1+r)^t + S - E1 - 0 - (S - E2) + 0$$

  $$PBT = (-C1 + P1 + C2 - P2)(1+r)^t + \mathrm{E2} - \mathrm{E1}$$

In the 3 cases, the PBT does not depend on S and is the unique.

4. Call put parity

Following a reverse conversion:

$PBT = (-C + P)(1 + r)^t - E + F$

Once arbitragers have benefited from this profit, which is the outcome of an abnormal discrepancy between the options and the futures markets, such a PBT is equal to 0.

Then:

$(-C + P)(1 + r)^t - E + F = 0$

Moreover, F is the best estimate of the future value of the underlying asset's current S_0 price.

In that case:

$(-C + P)(1 + r)^t - E + S_0(1 + r)^t = 0$

Dividend the above terms by $(1 + r)^t$:

$(-C + P) - E(1 + r)^{-t} + S_0 = 0$

Finally:

$P = C - S_0 + E(1 + r)^{-t}$

In continuous time:

$P = C - S_0 + E.e^{it}$

5. Cox-Ross-Rubinstein option pricing model

The stock price of the underlying asset can, at each period, either increase and go to *uS* (*u* as upside) at the end of the first period or decrease and go to *dS* (*d* as downside) with respective q and 1-q probabilities. In other terms, the rate of return of the stock over each period is either u-1 with probability q or d-1 with probability 1-q.

If the price of the underlying asset increases (or decreases), the premium of the call also increases (or decreases) with a probability q (or with probability1-q)

$$\begin{array}{ccc} & & uS \\ & \nearrow & \\ S & & \\ & \searrow & \\ & & dS \\ t=0 & & t=1 \end{array} \qquad\qquad \begin{array}{ccc} & & C_u \\ & \nearrow & \\ C & & \\ & \searrow & \\ & & C_d \\ t=0 & & t=1 \end{array}$$

The above trees mean that the expiration date is just one period away.

Moreover, on t=1, the value of the call corresponds to its pay off. Then:

C_u = max(0,uS-E) and C_d = max(0,dS-E).

Based on the forming of a hedging portfolio P which is made of the purchase of 1 call (premium: C) and the sale of H stocks: P = HS – C

As P is a hedging portfolio, it is riskless. Then its return corresponds to the risk free rate (r) and its value, at each period (hence when t=1), is unique:

$$HS–C = \frac{HuS - C_u}{1+r} = \frac{HdS - C_d}{1+r}$$

Then: $HuS–C_u = HdS – C_d$ and $HS = \dfrac{C_u - C_d}{u-d}$

Let $\hat{r}$ = 1+r. Then:

$HS–C = \dfrac{HuS - C_u}{1+r}$ and:

$$C = HS - \frac{HuS - C_u}{\hat{r}} = \frac{1}{\hat{r}}[\hat{r}\,HS - uHS + C_u]$$

$$= \frac{1}{\hat{r}}\left[\hat{r}\left(\frac{C_u - C_d}{u-d}\right) - u\left(\frac{C_u - C_d}{u-d}\right) + \left(\frac{u-d}{u-d}\right)C_u\right] = \frac{1}{\hat{r}}\left[\left(\frac{\hat{r}-d}{u-d}\right)C_u + \left(\frac{u-\hat{r}}{u-d}\right)C_d\right]$$

Let $p = \dfrac{\hat{r}-d}{u-d}$; then: $1-p = 1-\left(\dfrac{\hat{r}-d}{u-d}\right) = \dfrac{u-d-\hat{r}+d}{u-d} .= \dfrac{u-\hat{r}}{u-d}$

Finally : $C = \frac{1}{\hat{r}}.[pC_u+(1-p)C_d]$ [1]

The probability q does not appear in the formula. This means that even if different investors have different subjective probabilities about an upward or downward movement in the stock, they could agree on the relationship of C to S, u, d and r.

C does not depend on investors' attitudes towards risk either. The formula would therefore be the same whether investors are risk-averse or risk-preferring. Then, if investors are risk-neutral, the expected rate of return on the stock is the riskless interest rate, so:

$quS + (1-q)dS = \hat{r}\,S$ and $q=\frac{\hat{r}-d}{u-d}=p$

If the expiration date is 2 periods away:

$$
\begin{tikzcd}
 & & u^2S \\
 & uS \arrow[ur] \arrow[dr] & \\
S \arrow[ur] \arrow[dr] & & udS \\
 & dS \arrow[ur] \arrow[dr] & \\
 & & d^2S
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & C_{uu} \\
 & C_u \arrow[ur] \arrow[dr] & \\
C \arrow[ur] \arrow[dr] & & C_{ud} \\
 & C_d \arrow[ur] \arrow[dr] & \\
 & & C_{dd}
\end{tikzcd}
$$

t=0 t=1 t=2 t=0 t=1 t=2

In t = 1, the value of P is:

$$HuS-C_u = \frac{Hu^2S - C_{uu}}{\hat{r}} = \frac{HudS - C_{ud}}{\hat{r}}$$

Then: $Hu^2S-C_{uu} = HudS-C_{ud}$

and: $HuS = \frac{C_{uu} - C_{ud}}{u-d}$

$$HuS-C_u = \frac{Hu^2S - C_{uu}}{\hat{r}}$$

$$C_u = HuS - \frac{Hu^2S - C_{uu}}{\hat{r}}$$

$$C_u = \frac{1}{\hat{r}}[\,\hat{r}\,HuS - Hu^2S + C_{uu}] = \frac{1}{\hat{r}}[\,\hat{r}\,\frac{C_{uu} - C_{ud}}{u-d} - u\frac{C_{uu} - C_{ud}}{u-d} + \frac{u-d}{u-d}C_{uu}]$$

$$C_u = \frac{1}{\hat{r}}[\frac{\hat{r}-d}{u-d}C_{uu} + \frac{u-\hat{r}}{u-d}C_{ud}].$$

And : $C_u = \frac{1}{\hat{r}}[pC_{uu}+(1-p)C_{ud}]$ [2]

In the same way:

$$HdS–C_d = \frac{HudS - C_{ud}}{\hat{r}} = \frac{Hd^2S - C_{dd}}{\hat{r}}$$

Then : $HudS–C_{ud} = Hd^2S–C_{dd}$

Hence: $HdS = \frac{C_{ud} - C_{dd}}{u - d}$

And:

$$C_d = HdS - \frac{HudS - C_{ud}}{\hat{r}} = \frac{1}{\hat{r}}[\hat{r}\,HdS - HudS + C_{ud}]$$

$$= \frac{1}{\hat{r}}\left[\hat{r}\,\frac{C_{ud} - C_{dd}}{u - d} - u\,\frac{C_{ud} - C_{dd}}{u - d} + \frac{u-d}{u-d}C_{ud}\right]$$

$$= \frac{1}{\hat{r}}\left[\frac{\hat{r}-d}{u-d}C_{ud} + \frac{u-\hat{r}}{u-d}C_{dd}\right]$$

$$C_d = \frac{1}{\hat{r}}[pC_{ud}+(1-p)C_{dd}] \qquad [3]$$

If C_u and C_d are respectively replaced by [2] and [3] in [1]:

$$C = \frac{1}{\hat{r}}\{p\frac{1}{\hat{r}}[pC_{uu}+(1-p)C_{ud}]+(1-p)\frac{1}{\hat{r}}[pC_{ud}+(1-p)C_{dd}]\}$$

$$= \frac{1}{\hat{r}^2}\{p^2C_{uu}+p(1-p)C_{ud}+p(1-p)C_{ud}+p(1-p)C_{ud}+(1-p)^2C_{dd}\}$$

$$= \frac{1}{\hat{r}^2}[p^2C_{uu}+2p(1-p)C_{ud}+(1-p)^2C_{dd}]$$

If the expiration date is t = 2, the premium on t=2 corresponds to the call's payoff:

$$C = \frac{1}{\hat{r}^2}[p^2.\max(0, u^2S–E)+2p(1-p).\max(0, udS–E)+(1-p)^2.\max(0, d^2S–E)]$$

$$C = \frac{1}{\hat{r}^n}\sum_{k=0}^{n} Cn^k p^k(1-p)^{n-k}\max(0, u^k d^{n-k}S–E) \qquad [4]$$

If the expiration date is n periods away.

Let X be the random variable corresponding to the number of upward moves of the stock over n periods.

X is a binomial variable with n and p parameters: $X \mapsto B(n, p) \Rightarrow P[X = k] = C_n^k p^k(1-p)^{n-k}$

Let *a* stand for the minimal number of upward moves which the stock must make over the next n periods for the call to finish « in the money ». Then:

$$P[X \geq a] = \sum_{k=a}^{n} C_n^k\, p^k (1-p)^{n-k} = P[X=a]+P[X=a+1]+\ldots+P[X=n]$$

Moreover the value of C, given by [4] can be broken down into 2 terms:

$$C = \frac{1}{\hat{r}^n} \sum_{k=0}^{a-1} C_n^k\, p^k (1-p)^{n-k} \max(0, u^k d^{n-k} S - E)$$

$$+ \frac{1}{\hat{r}^n} \sum_{k=a}^{n} C_n^k\, p^k (1-p)^{n-k} \max(0, u^k d^{n-k} S - E)$$

If $0 < k < a$, the call is « out of the money and its payoff is null: $\max(0, u^k d^{n-k} S - E) = 0$

Otherwise : $\max(0, u^k d^{n-k} S - E) = u^k d^{n-k} S - E$.

And : $C = \frac{1}{\hat{r}^n} \sum_{k=a}^{n} C_n^k\, p^k (1-p)^{n-k} (u^k d^{n-k} S - E)$

$$C = \frac{1}{\hat{r}^n} \sum_{k=a}^{n} C_n^k\, p^k (1-p)^{n-k} u^k d^{n-k} S - E\,\hat{r}^{-n} \sum_{k=a}^{n} C_n^k\, p^k (1-p)^{n-k}$$

$$C = S \sum_{k=a}^{n} C_n^k \left(\frac{up}{\hat{r}}\right)^k \left[\frac{d(1-p)}{\hat{r}}\right]^{n-k} - E\,\hat{r}^{-n} \sum_{k=a}^{n} C_n^k\, p^k (1-p)^{n-k}$$

Let: $p' = \frac{up}{\hat{r}}$. Then : $1-p' = 1-\left(\frac{u}{\hat{r}}\right)p = 1-\left(\frac{u}{\hat{r}}\right)\left(\frac{\hat{r}-d}{u-d}\right)$

$$= \frac{\hat{r}(u-d)-u(\hat{r}-d)}{\hat{r}(u-d)}$$

$$= \frac{d(u-\hat{r})}{\hat{r}(u-d)} = \frac{d}{\hat{r}}(1-p)$$

And finally : $C = S \sum_{k=a}^{n} C_n^k\, p'^k (1-p')^{n-k} - E\,\hat{r}^{-n} \sum_{k=a}^{n} C_n^k\, p^k (1-p)^{n-k}$

Let F (a,n,p) be the complementary binomial distribution function :

$$C = S F(a,n,p') - E\,\hat{r}^{-n} F(a,n,p)$$

If *n* - ie the number of periods (or subintervals) between the valuation date and the expiration date - is very high, the multiplicative binomial distribution of stock prices goes to the lognormal distribution and the CRR formula converges towards the Black & Scholes one:

$$C = S\Phi(d_1) - Ee^{-r'\tau}\Phi(d_2) \text{ with } d_1 = \frac{\ln\frac{S}{E} + (r' + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}},\ d_2 = d_1 - \sigma\sqrt{\tau} \text{ and } \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt$$