Practical use of the Cox Ross and Rubinstein (CRR) formula

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1. Common parameters with the Black and Scholes formula

S = spot price of the underlying asset = 120 €
E = strike price of the call = 100 €
σ = volatility of the underlying asset = 40%
r = discrete risk-free rate = 2%
τ = time to expiration in year(s) = 0,36 based on the following dates
Valuation date: 06/11/2019
Expiration date: 15/03/2019
Number of remaining days till expiration date = 130
Number of remaining year(s) till expiration date = $\tau = \frac{130}{365} = 0{,}36$

2. Specific additional parameters for the CRR formula

a. Formulas

$u = e^{\sigma\sqrt{\frac{\tau}{n}}}$ = upward move
$d = \frac{1}{u}$ = downward move

n = number of periods dividing the time before the valuation date and the maturity date

$p = \frac{\hat{r}-d}{u-d}$ = probability of an upward move during a period

$1 - p = \frac{u-\hat{r}}{u-d}$ = probability of a downward move during a period

with $\hat{r} = 1 + r$ assuming 1 period = 1 year.

Assuming a more general context:
- time to expiration, in years = τ
- number of periods = *n*

then:
- 1 period = $\frac{\tau}{n}$ year
- $\hat{r} = (1 + r)^{\frac{\tau}{n}}$

b. Numerical applications

$u = e^{0,4\sqrt{\frac{0,36}{10}}} = 1,08$

$d = \frac{1}{1,08} = 0,93$

$n = 10$

$\hat{r} = (1 + 2\%)^{\frac{0,36}{10}} = 1,02^{0,036}$

$p = \frac{\hat{r} - d}{u - d} = \frac{1,02^{0,036} - 0,93}{1,08 - 0,93} = 0,49$

$1 - p = 0,51$

This set of numerical assumptions enables to build a tree corresponding to the possible evolutions of the underlying asset's spot price (S) from the valuation date (06/11/2019) to the expiration date of the call (15/03/2020).
This time, made of 130 days (calculation performed by Excel: 15/03/2020 – 06/11/2019=130) or τ = 0,36 year, can be broken down into 10 periods. In that case, each period is equal to 0,036 year.
The valuation date corresponds to the period the reference of which is 0; the expiration date corresponds to the end of the 10th period.

On valuation date, S is worth 120 €. At the end of the 1st period, it becomes:
– either: $1,08x120 = 129$ € assuming an upward move;
– or: $0,93x120 = 111$ € assuming a downward move.

This is represented by:

		129
S=	120	
		111
t=	0	1

On t=1:
– if S_1 is worth 129 €, it becomes, at the end of 2nd period:
  - either: $1,08x129 = 140$ € assuming a new upward move;
  - or: $0,93x129 = 120$ € assuming a downward move;
– if S_1 is worth 111 €, it becomes, at the end of 2nd period:
  - either: $1,08x111 = 120$ € assuming an upward move;
  - or: $0,93x111 = 103$ € assuming a new downward move…

			140
		129	0
S=	120		120
		111	0
			103
t=	0	1	2

Etc…

Excel enables to extend the formulas, on the right, then on the top, then on the bottom:

	0	0	0	0	0	0	0	0	0	255	
	0	0	0	0	0	0	0	0	237	0	
	0	0	0	0	0	0	0	220	0	220	
	0	0	0	0	0	0	204	0	204	0	
	0	0	0	0	0	189	0	189	0	189	
	0	0	0	0	175	0	175	0	175	0	
	0	0	0	162	0	162	0	162	0	162	
	0	0	150	0	150	0	150	0	150	0	
	0	140	0	140	0	140	0	140	0	140	
	129	0	129	0	129	0	129	0	129	0	
120		120	0	120	0	120	0	120	0	120	
	111	0	111	0	111	0	111	0	111	0	
	0	103	0	103	0	103	0	103	0	103	
	0	0	96	0	96	0	96	0	96	0	
	0	0	0	89	0	89	0	89	0	89	
	0	0	0	0	82	0	82	0	82	0	
	0	0	0	0	0	76	0	76	0	76	
	0	0	0	0	0	0	71	0	71	0	
	0	0	0	0	0	0	0	66	0	66	
	0	0	0	0	0	0	0	0	61	0	
	0	0	0	0	0	0	0	0	0	56	
0	1	2	3	4	5	6	7	8	9	10	= expiration date

Getting this tree has consisted in moving forward in time. Hence the direction of the blue arrow.

The second tree, which is dedicated to the evolution of the call's premium will consist in going back in time.
At t=10, ie on expiration date, the call's premium is equal to its intrinsic value. For each possible value of S_{10} in the last column of the first tree, a corresponding call's intrinsic value can be calculated:
$C_{10} = \max(0, S_{10} - E) = \max(0, S_{10} - 100)$
Then:

- For $S_{10} = 255, C_{10} = 155$
- For $S_{10} = 220, C_{10} = 120$
- …
- For $S_{10} < 100, C_{10} = 0$

Moving backward to t=9:

- If, on t=10, $C_{10} = 155$, the calculation of C_9 is based on:

$$C = \frac{1}{\tilde{r}}[pC_u + (1-p)C_d]$$

Where:

- C_u = call's premium posted at the end of the next period assuming an upward move ie 155 €
- C_d = call's premium posted at the end of the next period assuming a downward move ie 120 €

Then: $C_9 = \frac{1}{1{,}02^{0{,}036}}[0{,}49x155 + 0{,}51x120] = 137$ €

– If, on t=10, $C_{10} = 120$, $C_9 = \frac{1}{1{,}02^{0{,}036}}[0{,}49x89 + 0{,}51x120] = 104$ €
– …

Excel enables to extend the formulas, on the left, then on the top, then on the bottom:

											155	
										137	0	
									120	0	120	
								104	0	104	0	
							89	0	89	0	89	
						75	0	75	0	75	0	
					63	0	63	0	62	0	62	
				51	0	51	0	51	0	51	0	
			41	0	40	0	40	0	40	0	40	
		32	0	31	0	30	0	30	0	29	0	
C=	**23,86**	0	23	0	22	0	21	0	20	0	20	
		17	0	16	0	14	0	13	0	11	0	
			11	0	9,5	0	8,1	0	6,3	0	3,2	
				6,1	0	4,9	0	3,4	0	1,5	0	
					2,9	0	1,9	0	0,8	0	0	
						1	0	0,4	0	0	0	
							0,2	0	0	0	0	
								0	0	0	0	
									0	0	0	
										0	0	
											0	
	0	1	2	3	4	5	6	7	8	9	10	= expiratior

Going on going backward, on t=0, $C_0 = 23{,}86$ €

A Black and Scholes' pricing model provides the same output.